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1981 J. Phys. A: Math. Gen. 14 2459

(http://iopscience.iop.org/0305-4470/14/9/038)

J. Phys. A: Math. Gen. **14** (1981) 2459–2465. Printed in Great Britain

The Umezawa–Visconti relation for first-order field equations

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Received 29 January 1981

Abstract. We discuss the Umezawa–Visconti relationship between nilpotency index q and spin s for first-order equations derivable from a Lagrangian, and argue that previous investigations have been too pessimistic. Graph theoretical bounds are given for q, which improve on previous estimates and give $q \leqslant 2s-1$ directly for the Singh–Hagen equations. It is argued that the Hurley equations cannot be extended to a Lagrangian system while maintaining $q=1$, contrary to previous claims.

1. Introduction

Recently Mathews *et al* (1980b) have discussed the Umezawa–Visconti (UV) result (Umezawa and Visconti 1956). This result claims that the nilpotency index q in the minimal equation

$$(L_0)^{q+2} = (L_0)^q \tag{1.1}$$

for the manifestly Lorentz covariant first-order equation

$$(-\mathrm{i}L_\mu \partial^\mu + m)\psi = 0 \tag{1.2}$$

for a unique mass-m, spin-j_{p} field is given by $q = 2j_{\mathrm{p}} - 1$. Mathews *et al* rightly observe that there has been some confusion over this result, and that while it has been known for some time to be not always correct (Glass 1971), the precise status of a relation between q and j_{p} has never been satisfactorily resolved. Various gross exceptions to the result, such as the Hurley equations (Hurley 1971), for which $q=1$ for *all* spins, have gradually undermined confidence in its validity. This is a pity, because a relation between q and j_{p} is a good practical guide to what sort of free field theories are available for a given spin, and also to the complexity of the constraint structure, which bears on the various interaction difficulties of an equation (Mathews *et al* 1980a).

However, the analysis of Mathews *et al* (1980b) is unduly pessimistic. They give no lower bound on q, citing the Hurley equations as evidence for the absence of such a bound. In fact, for truly self-contained first-order field theories (the Λ theories referred to below) derived from a Lagrangian there is a non-trivial lower bound, which we will discuss in § 2, where we also discuss the claim of Mathews *et al* (1980b) that the Hurley equation may be extended to a Lagrangian system without increasing q. The upper bounds for q obtained by Mathews *et al* depend only on the *number* of Lorentz irreducible representations (irreps) carried by ψ which contribute to a particular spin.

This in general takes no account of the connectivities between these irreps implied by the non-zero elements of L_0—and these clearly have a bearing on the minimal equation (1.1). Only in the case of the Singh–Hagen equations (Singh and Hagen 1974a, b) for integer spin is the more detailed analysis presented, yielding the result $q \leq 2j_p - 1$ only after elementary row operations. In § 3 we extend to the integer-spin case a graphical technique for finding upper (and lower) bounds on q, previously given by the author (Cox 1978) for half-odd-integer spin. This technique does take account of the connectivities between the irreps in R, and consequently is often a good improvement on the bounds of Mathews *et al* (1980b) obtained by *counting* irreps. In particular, it gives $q \leq 2s - 1$ for the Singh–Hagen boson spin-s equations directly, without the need for elementary row operations.

In the following we omit details of some proofs, as the ideas and notation are obvious extensions of those contained in the earlier paper (Cox 1978).

2. The lower bound on q

First, we need an unambiguous way of specifying the particular theory of form (1.2) which we have in mind. For this, we need only specify the representation, R, of the proper Lorentz group L_p, carried by ψ, and display L_0 in a convenient representation (the L_i are then determined by a Lorentz transformation in the space of R). It is convenient to depict such a theory by a graph whose vertices correspond to the irreps in ψ and in which two vertices are joined by a directed edge if the corresponding elements of L_0 are non-zero. The resulting graph is a pictorial representation of the so-called 'skeleton matrix' of Mathews *et al* (1980b). For notation, terminology and further details we refer to the previous papers (Cox 1974a, b, c, 1978).

In the case of non-zero mass, $m \neq 0$, L_0 determines the mass–spin spectra in a well known way. L_0 commutes with the rotation group generators and so can be reduced to block diagonal form in which each block corresponds to a different irrep of the rotation group contained in R—the so-called spin-blocks. The graphs for these s-blocks are obtained by taking only those vertices corresponding to L_p irreps in R which contain the irrep $D(s)$ in their rotation group decomposition. The eigenvector corresponding to a non-zero eigenvalue, λ, of the s-block describes a field with spin s and mass m/λ. For simplicity we will here only consider unique mass and spin theories, for which all s-blocks are nilpotent, except that corresponding to the required physical spin $s = j_p$, which must have precisely two non-zero eigenvalues $\pm\lambda$. The extension of the entire discussion to the multi mass–spin case is purely a matter of organisation (see Cox (1978) for the half-odd-integer spin case).

Umezawa and Visconti (1956, Takahashi 1969) gave an argument to show that

$$q + 2 = 2j_p + 1. \tag{2.1}$$

This argument involves separately showing that $q + 2 \geq 2j_p + 1$ and $q + 2 \leq 2j_p + 1$. Glass (1971) showed that their analysis of the transformation properties of the L_μ algebra had been insufficiently rigorous, and that in general $q + 2 \leq 2j_p + 1$ did not hold. Glass did not put an upper bound on q, although he gave the correct algebraic constraints on the L_μ algebra which effectively replace the UV bound. There is one obvious upper bound on q, provided by the size of the largest s-block, which can be improved upon when various symmetries such as reflection covariance are demanded. This is the sort of bound used by Mathews *et al*, and in § 3 we will show how to improve on this by simple graphical techniques. It is important to realise that Glass did not quarrel with the *lower* bound, $q+2 \geqslant 2j_{\mathrm{p}}+1$, to which we now turn.

Along with the work of Glass, a number of authors found theories for which $q+2<2j_{\mathrm{p}}+1$ (Hurley 1971, Chandrasekaran *et al* 1972), and the belief grew that in fact the UV result had very little content left at all. Attempts to salvage the relation (Santhanam and Tekumalla 1974) were not altogether convincing because the basic assumptions of the theory were not made clear, and tended to confuse the matter. Chandrasekaran *et al* (1972) modified the UV relation to

$$2j_{\mathrm{p}}+1 \leqslant q+2 \leqslant 2j_{\mathrm{m}}+1 \tag{2.2}$$

where j_{p} is as before the maximum total physical spin of the field, projected out of ψ by the field equation (1.2), and j_{m} is the maximum spin, or weight, in the rotation group decomposition of $\boldsymbol{R}$. Umezawa and Visconti had not taken pains to distinguish between j_{p} and j_{m}, because it seems quite natural to take $j_{\mathrm{p}}=j_{\mathrm{m}}$—why introduce higher spins in ψ which only have to be knocked out by more troublesome constraints? Now, however, such possibilities must be considered. For example, the only known causal theory with $q>1$ is the spin-1 theory of Capri and Shamaly (1973) in which ψ contains up to spin 2.

It was the work of Capri (1969) which raised the problem of a distinction between j_{p} and j_{m}—he was looking for alternative equations for the electron in which non-physical spin-$\frac{3}{2}$ fields were introduced. The work of Chandrasekaran *et al* (1972) and Santhanam and Tekumalla (1974) was partly in response to Capri's apparent violation of the UV relation. These relations (2.2) obtained by these authors were partly incorrect and partly incomplete. Their argument for $q+2 \leqslant 2j_{\mathrm{m}}+1$ is the same as that of Umezawa and Visconti, and fails for the same lack of rigour as noted by Glass—the point about Glass's arguments is that we *can* have $q+2>2j_{\mathrm{m}}+1$. The argument for $q+2>2j_{\mathrm{p}}+1$ is also essentially that of Umezawa and Visconti, but with the observation that the L_μ algebra must provide at least the *physical* degrees of freedom in the theory, which simply amount to $2j_{\mathrm{p}}+1$ ($\times 2$ for the particle–antiparticle pair). Now this result is quite correct, as was the original argument of Umezawa and Visconti, provided we are dealing with theories derivable from a Lagrangian and possessing a hermitising matrix Λ such that

$$\Lambda L_\mu^{+}\Lambda = L_\mu.$$

We will call such theories Λ theories. The UV argument for $q+2>2j_{\mathrm{p}}+1$ assumes a Λ theory, because only then does the Klein–Gordon divisor exist with the required properties (Takahashi 1969). Indeed, to do justice to Umezawa and Visconti, with their assumptions of existence of Λ, and the Klein–Gordon divisor a polynomial in $L\cdot\partial$, and $j_{\mathrm{p}}=j_{\mathrm{m}}$, their result is quite correct. It is for theories violating one or other of these conditions that the result understandably breaks down. Thus, the Hurley equations have no local Λ (one can be constructed using derivatives (Hurley 1974), but the resulting non-local relation between the field and its dual take us too far away from conventional quantum field theory, for most tastes) and are not directly derivable from a Lagrangian without the introduction of further auxiliary fields. This of course introduces a completely new set of equations, with a new L_0, and it is this new L_0 which will be subject to $q+2 \geqslant 2j_{\mathrm{p}}+1$.

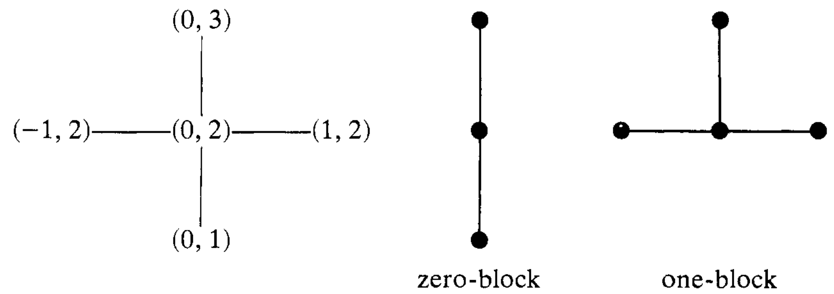
Mathews *et al* (1980b) have claimed that one can extend the Hurley equations to a Lagrangian system without increasing q. However, they did not show how this might be done, and in fact we will argue below that it does not seem possible.

First, however, we give an heuristic argument suggesting that for Λ theories there has to be a lower bound on q which is connected to the physical spin. In general, for irreducible Λ-theories, the higher the spin j_p, the more irreps of L_p must be contained in R to provide an irreducible system of equations—essentially these extra representations are to link the field with its conjugate to provide an Hermitian form. So the higher the spin, the larger is the representation space R, and so the larger the representation of the L_μ required. This means that the number of independent elements in the L_μ algebra must be correspondingly larger, and since there are only four different L_μ (i.e. four generating elements of the L_μ algebra) this can only mean that the degree of the equations satisfied by the L_μ must increase. Thus higher j_p implies higher q. The UV lower limit quantifies this inevitable relation between q and j_p.

Now even for non Λ-theories there is also an unavoidable lower bound on q, obvious from the graph of the theory concerned. For every pair of vertices in an s-block graph, consider the paths of minimum length connecting them. For some pairs of vertices there will be precisely one such minimum path. Let $d_u^{(s)}$ be the length of the longest of these 'unique' paths. Then the degree of nilpotency of that block must exceed $d_u^{(s)}$ (Cox 1978), so the degree of nilpotency for L_0 must exceed $\sup d_u^{(s)} = d_u$, i.e. $q \geqslant d_u + 1$. We can use this almost trivial result to investigate the claim of Mathews *et al* (1980b) that the Hurley equations can be extended to a Lagrangian system without increasing q.

For spin j_p the graph of the Hurley equation is simply one edge, connecting the irreps $D(j_p, 0)$ and $D(j_p - \frac{1}{2}, \frac{1}{2})$. To construct a conventional Lagrangian, we have at least to introduce the conjugates of these representations, $D(0, j_p)$ and $D(\frac{1}{2}, j_p - \frac{1}{2})$ and then link these conjugates to the originals in some way, using further irreps, to ensure a system of equations which is not covariantly reducible to two separate systems. This inevitably, for spin > 1, introduces graphs contributing to spin blocks for $s < j_p$. These lower spin blocks must be nilpotent and at least one will be non-zero, and so have a minimum polynomial of the form $A_s^{q_s} = 0$, $q_s > 1$. This means that q will be >1. So when we extend the Hurley equations to a Λ theory we must have at least $q > 1$, contrary to the claim of Mathew *et al* (1980b). In fact, when one tries to construct a Λ theory starting as above, by connecting the conjugate irreps, while retaining the required mass–spin spectra and a non-zero charge energy density, it becomes apparent that values of the order of $2s$ will be generated for $d_u^{(s)}$, with corresponding implications for q. We have found no contradiction to the Umezawa–Visconti result $q \geqslant 2j_p - 1$.

The above graphical lower bound is in some cases obviously nothing like as good as $q + 2 \geqslant 2s + 1$ for Λ theories, but in others it is a welcome improvement. For example, consider the spin-zero theory based on the graph



It can easily be shown that the one-block can be made nilpotent, with $q = 3$, and the zero-block produces a mass. So in this case $q = 2(2) - 1 = 2j_m - 1$ and the UV relation

$q+2 \geqslant 2j_{\mathrm{p}}+1=1$ is rather inadequate. However, from the graph of the one-block $d_{\mathrm{u}}=2$, and so $q \geqslant d_{\mathrm{u}}+1$ gives $q \geqslant 3$. Incidentally, the maximal matching method described in the next section gives $q \leqslant 2(1)+1=3$, since for the one-block $\beta_1=1$. So in this case graphical considerations alone correctly give $q=3$. We have to admit that it is not so successful for higher spins, but nevertheless it still improves on previous estimates.

3. The upper bound on q

The UV upper bound is violated not because their arguments were incorrect, but because their assumption about the Klein–Gordon division was too restrictive. They assumed that it was a polynomial in $L \cdot \partial$, when in general it need not be (Glass 1971, Santhanam and Tekumalla 1974). In a previous paper (Cox 1978) the author has given a graphical method for obtaining an upper bound on q, for the case of half-odd-integer spin Λ-theories with parity invariance. For any given nilpotent s-block, the degree of nilpotency q_s must for half-odd-integer spin Λ-theories with parity invariance satisfy

$$q_s \leqslant \begin{cases} \beta_1^{(s)}+1, & \\ \beta_1^{(s)} & \text{(if the maximal matching is perfect)} \end{cases}$$

where $\beta_1^{(s)}$ is the number of edges in a maximal matching (a maximum set of mutually disjoint edges in the graph) on the graph of the s-block. (A perfect matching is one which covers all vertices of the graph.) Then q, the nilpotency index for L_0, is bounded above by sup $(\beta_1^{(s)}+1)$ or sup $\beta_1^{(s)}$. This bound on $q^{(s)}$ is clearly never worse than that of Mathews *et al*, namely $\frac{1}{2}l_s$ where l_s is the number of irreps entering the s-block, and is in many cases an improvement.

For the integer-spin case a slight modification is necessary. We can still number the irreps in R to write any s-block in the form

$$A_s=\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix},$$

but whereas in the half-odd-integer spin Λ-theories with parity invariance we had $A=B$, in the case of similar integer-spin theories A and B may not even be of the same size and are unequal in general. However, it is still true that $\operatorname{rank}(A)=\operatorname{rank}(B)$ and that for the corresponding graph of the s-block we have

$$\operatorname{rank}(A) \leqslant \beta_1^{(s)}.$$

Hence

$$\operatorname{rank}(A_s)=2 \operatorname{rank}(A) \leqslant 2\beta_1^{(s)}$$

and if A_s is nilpotent then

$$q_s \leqslant \operatorname{rank}(A_s)+1$$

so

$$q_s \leqslant 2\beta_1^{(s)}+1.$$

Again, if the matching is perfect then $2\beta_1^{(s)}$ is simply the size of A_s and so clearly

$$q_s \leqslant 2\beta_1^{(s)} \qquad \text{if matching perfect.}$$

So, for integer-spin s-blocks of Λ theories

$$q_s \leqslant 2\beta_1^{(s)}+1 \qquad \text{if MM not perfect,}$$
$$\leqslant 2\beta_1^{(s)} \qquad \text{if it is.}$$

This bound appears to be 'twice as bad' as that for the Fermi case, because of the factor two, but because of the structure of the graphs for integer spin this turns out not to be the case. In fact, for both Fermi (Cox 1978) and, as we now show, Bose Singh–Hagen theories it gives directly the same result $q \leqslant 2s-1$.

By inspection of the Singh–Hagen first-order equations for integer spin (Singh and Hagen 1974a), the graph of the theory for spin s is as shown in figure 1, along with the zero- and one-blocks for the theory, which will provide the best bounds for q.

For the zero-block a little consideration shows that $d_u = s+1$ and $\beta_1 = s-1$, so for this block

$$s+2 \leqslant q_0 \leqslant 2s-1.$$

Similarly, for the one-block $d_u = s+1$ and $\beta_1 = s-1$, so also

$$s+2 \leqslant q_1 \leqslant 2s-1.$$

Thus, by direct visual inspection of the graphs we easily obtain $q \leqslant 2s-1$ for this theory. There is no need for elementary row and column operations or other calculations of rank, as done by Mathews *et al* (1980b). Also we see that it is not sufficient merely to

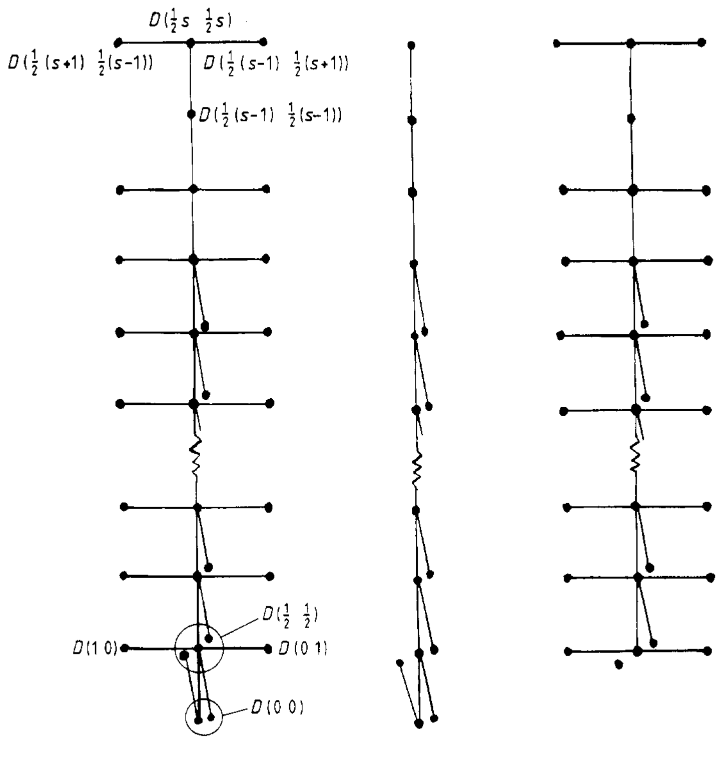


Figure 1. Graph of the Singh–Hagen equation for integer spin s, with the zero- and one-blocks.

consider the *largest* s-block (in this case the one-block) as done by Mathews *et al*, for here a smaller block (the zero-block) gives the same result.

Note that in this theory the lower, d_u+1, bound is rather weak for $s>3$. The Singh–Hagen equations come from a Λ theory and so the UV lower bound $q \geqslant 2s-1$ holds, and with the above graphical upper bound we thus conclude that $q=2s-1$, as in the half-odd-integer spin case.

4. Conclusion

For first-order theories derivable from a Lagrangian, the UV relationship between nilpotency index and spin is in general more complicated than originally thought, but nevertheless can be replaced by a systematic graph theoretical procedure which improves on previous methods. The lower estimate $q \geqslant 2j_p - 1$ remains, and can sometimes be improved upon by a simple graphical result $q \geqslant d_u + 1$. For the upper limit one need only look at the maximal matchings on the s-block graphs to obtain upper limits for their nilpotency indices and thus for the overall nilpotency index q. The results for half-odd-integer spin have been given previously (Cox 1978), and for integer spin have here been extended to

$$\begin{aligned} q^{(s)} &\leqslant 2\beta_1^{(s)}+1 \qquad &\text{(non-perfect MM)},\\ &\leqslant 2\beta_1^{(s)} \qquad &\text{(perfect MM)}. \end{aligned}$$

Also note that these same results apply to *non-parity invariant* half-odd-integer spin theories also.

These graphical bounds give immediately $q=2s-1$ for the spin-s Singh–Hagen Bose equations.

Finally, in the previous paper (Cox 1978) we showed how for Fermi theories q could be increased past $2j_m-1$ by adding additional copies of lower irreps of L_p, so that the spin j_m did not increase, but β_1 did. Similar tricks can be used in the integer-spin case also.

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